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On third-order radiation theory for axisymmetric bodies undergoing heave motion of multiple frequencies

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Summary.

As a first step in the development of a nonlinear theory for calculating the response of an axisymmetric system to an irregular sea, the radiation problem for axisymmetric floating or immersed bodies in a periodic heave motion, composed of a number of harmonic components, is considered by means of a third-order potential theory.

It is shown that the knowledge of only first- and second-order potential functions is required for the calculation of all forces up to the third order. A boundary integral equation method is proposed for the determination of these potential functions.

1. Introduction

A linear approach to hydrodynamic problems concerning the behaviour of a floating or submerged body in an irregular seaway can result in realistic prediction if motion and wave amplitudes are small. If they are not, for instance when wave-power absorption by floating bodies is considered, nonlinearities have to be taken into account, especially when the hull of the body is not vertical near the waterline, so that nonlinear hydrostatic forces are induced.

The development of a second-order theory seemed to be insufficient, because in the special case of a harmonic motion second-order forces do not influence the first harmonic of the force, thus the power absorption. Moreover, the wave-energy project of the Office of Naval Architecture (State University Ghent, Belgium), in which the author is involved, considers the case of an axisymmetric body with a conical shape near the waterline, which implies that restoring forces contain a cubic term (Ferdinande [2], [3]; Ferdinande and Vantorre [5]).

A first step in the development of a nonlinear theory for calculating the response of an axisymmetric system to irregular waves is presented in this paper. A third-order potential theory has been developed for calculating forces acting on axisymmetric floating or immersed bodies in periodic heave motion, composed of a number of harmonic components. The nonlinear Bernoulli equation is used and boundary conditions are fulfilled on the instantaneous body and free water surfaces. The vertical force is calculated by

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integration of the pressure over the instantaneous body-surface position, and can be written as the sum of a number of first-, second- and third-order terms. The linear terms can be expressed in a similar way as the motion; the second- and third-order forces are composed of harmonic components of which the frequencies are sums and differences of combinations of two, respectively three, frequencies of the motion harmonics.

A reduction of the order of the problem has been obtained by extending a method developed by Söding [7]. The knowledge of first- and second-order potential functions is sufficient to calculate all forces up to the third order. For the calculation of these potentials, a boundary integral equation method developed by Kritis [6] (see also Ferdinande and Kritis [4]), based on Yeung's work [10] (see also Bai and Yeung [1]), has been extended.

A more detailed description of this theory is available in Dutch (Vantorre [8]). The radiation problem for an axisymmetric body in forced harmonic heave motion of a single frequency was considered by Vantorre [9].

2. Boundary conditions

The floating or submerged axisymmetric body is assumed to undergo a periodic heave motion, expressed as

$$\zeta(t) = \sum_{j=1}^{N} \mathrm{Re}\left(\bar{\zeta}_j \, e^{i\omega_j t}\right) \tag{1}$$

where $\bar{\zeta}_j$ is the complex amplitude of motion for frequency ω_j. Two polar coordinate systems are used (see Figure 1): a fixed system (x, z, θ), and a system $(\bar{x}, \bar{z}, \bar{\theta})$ fixed to the body, so

$$x = \bar{x}, \tag{2}$$

$$z = \bar{z} + \zeta, \tag{3}$$

$$\theta = \bar{\theta}. \tag{4}$$

A local coordinate system (s, n) can be considered in each point of the body; the angle between the s-axis and the z-axis is indicated by α. l is the contour length, measured along the intersection of the body with the $0xz$-plane. Differentiation with respect to l is denoted by a prime ($'$). The wave elevation in a point of the free surface is denoted by Z. If the motion of the fluid is assumed to be irrotational, a velocity-potential function Φ can be defined:

$$\mathbf{v} = \nabla\Phi \tag{5}$$

with $\mathbf{v}$ denoting the velocity of the fluid particles. Laplace's equation has to be satisfied,

$$\Phi_{xx} + \frac{1}{x}\Phi_x + \Phi_{zz} = 0, \tag{6}$$

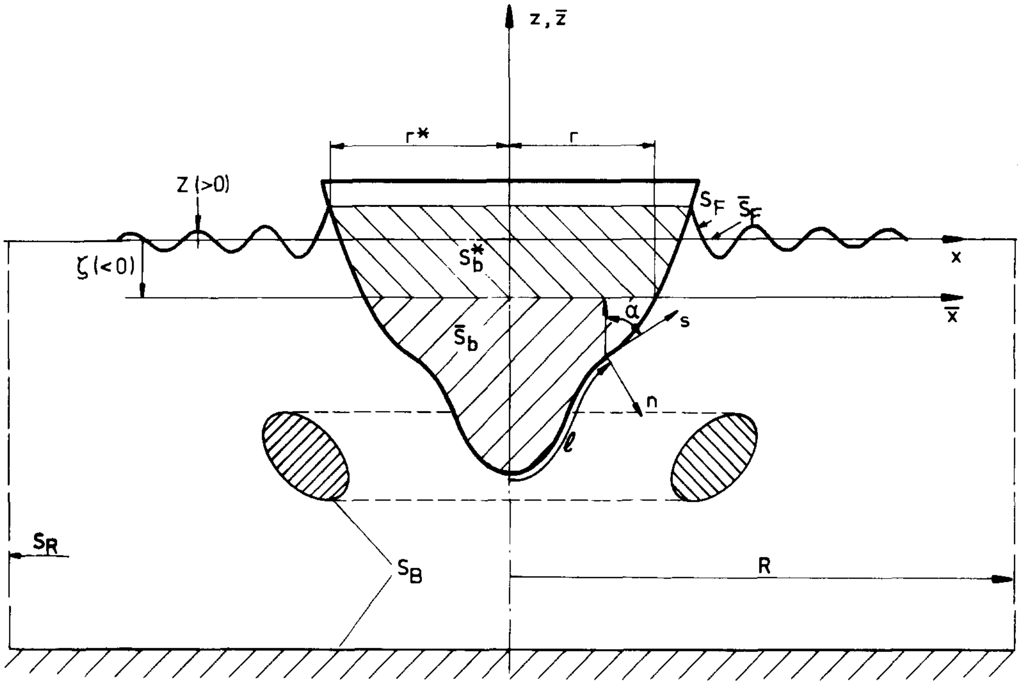

Figure 1.

together with the following boundary conditions:

(i) On the free surface, S_F, the pressure p does not change with time and is equal to the atmospheric pressure p_0. Using Bernoulli's equation, and expanding the potential function in a Taylor series in order to obtain an expression on the mean position $\bar{S}_F$ of the free surface, this condition can be written as

$$\begin{aligned}
&\Phi_{tt} + g\Phi_z + 2\Phi_x\Phi_{xt} + 2\Phi_z\Phi_{zt} - \Phi_{zz}\Phi_t - \frac{1}{g}\Phi_t\Phi_{ztt} + \Phi_x^2\Phi_{xx} + 2\Phi_x\Phi_z\Phi_{xz} + \Phi_z^2\Phi_{zz} \\
&\quad - \frac{1}{2g}\left(\Phi_x^2 + \Phi_z^2\right)\Phi_{ztt} + \frac{1}{2g^2}\Phi_t^2\Phi_{zztt} - \tfrac{1}{2}\left(\Phi_x^2 + \Phi_z^2\right)\Phi_{zz} + \frac{1}{2g}\Phi_t^2\Phi_{zzz} - \frac{2}{g}\Phi_x\Phi_t\Phi_{xzt} \\
&\quad - \frac{2}{g}\Phi_t\Phi_{xt}\Phi_{xz} - \frac{2}{g}\Phi_z\Phi_t\Phi_{zzt} - \frac{1}{g}\Phi_t\Phi_{zt}\Phi_{zz} + \frac{1}{g^2}\Phi_t\Phi_{zt}\Phi_{ztt} = 0 \qquad \text{on} \quad \bar{S}_F(z=0);
\end{aligned} \tag{7}$$

(ii) Normal velocities of fluid particles and body have to be equal at the wetted body surface S_b:

$$-\dot{\zeta}\sin\alpha = \Phi_n \qquad \text{on} \quad S_b, \tag{8}$$

$$\begin{aligned}
&= \Phi_n - \zeta\Phi_{nn}\sin\alpha + \zeta\Phi_{ns}\cos\alpha \\
&\quad + \tfrac{1}{2}\zeta^2\Phi_{nnn}\sin^2\alpha - \zeta^2\Phi_{nns}\cos\alpha\sin\alpha \\
&\quad + \tfrac{1}{2}\zeta^2\Phi_{nss}\cos^2\alpha \qquad \text{on} \quad \bar{S}_b;
\end{aligned} \tag{9}$$

(iii) On the bottom and on other fixed axisymmetric bodies in the fluid, the normal velocity component equals zero:

$$\Phi_n = 0 \qquad \text{on} \quad S_B; \tag{10}$$

(iv) Radiation conditions will be developed in the sequel.
Note: In expressions (7) and (9) terms of fourth or higher order have been neglected.

3. Splitting the potential function

The potential function is now written as the sum of first-, second- and third-order terms,

$$\Phi = {}_1\Phi + {}_2\Phi + {}_3\Phi. \tag{11}$$

If (11) is substituted into (6), (7), (9) and (10), and first-, second- and third-order terms are grouped, conditions for ${}_1\Phi$, ${}_2\Phi$ and ${}_3\Phi$ can be found.

The set of boundary conditions for the first-order potential function ${}_1\Phi$ reveals an expression for ${}_1\Phi$:

$${}_1\Phi = \sum_{j=1}^{N} \mathrm{Re}\left[{}_1\bar{\Phi}^{(\omega_j)}\, \mathrm{e}^{i\omega_j t}\right] = \sum_{j=1}^{N} \mathrm{Re}\left[\bar{\zeta}_j\, {}_1\bar{\phi}^{(\omega_j)}\, \mathrm{e}^{i\omega_j t}\right] \tag{12}$$

where

$${}_1\bar{\Phi}^{(\omega_j)} = \bar{\zeta}_j\, {}_1\bar{\phi}^{(\omega_j)}. \tag{13}$$

Nonhomogeneous terms of the boundary conditions of ${}_2\Phi$ are composed of products of two first-order terms (ζ, ${}_1\Phi$). It can be shown that ${}_2\Phi$ has to take the following form:

$$\begin{aligned} {}_2\Phi &= \sum_{\substack{k=-N \\ k\neq 0}}^{N} \sum_{j=|k|}^{N} \mathrm{Re}\left[{}_2\bar{\Phi}^{(\omega_j+\omega_k)}\, \mathrm{e}^{i(\omega_j+\omega_k)t}\right] \\ &= 2 \sum_{\substack{k=-N \\ k\neq 0}}^{N} \sum_{j=|k|}^{N} \mathrm{Re}\left[A_{jk}\bar{\zeta}_j\bar{\zeta}_k\, {}_2\bar{\phi}^{(\omega_j+\omega_k)}\, \mathrm{e}^{i(\omega_j+\omega_k)t}\right] \end{aligned} \tag{14}$$

where

$${}_2\bar{\Phi}^{(\omega_j+\omega_k)} = 2A_{jk}\bar{\zeta}_j\bar{\zeta}_k\, {}_2\bar{\phi}^{(\omega_j+\omega_k)}, \tag{15}$$

$$\begin{aligned} A_{jk} &= \tfrac{1}{2} \qquad \text{if} \quad |j| = |k|, \\ &= 1 \qquad \text{if} \quad |j| \neq |k|, \end{aligned} \tag{16}$$

$$\omega_{-j} = -\omega_j, \tag{17}$$

$$\bar{\zeta}_{-j} = \bar{\zeta}_j^* \qquad \text{(complex conjugate value)}. \tag{18}$$

Nonhomogeneous terms of the boundary conditions of ${}_3\Phi$ are composed of (a) products of three first-order terms (ζ, ${}_1\Phi$), and (b) products of a first-order term (ζ, ${}_1\Phi$) with a second-order term (${}_2\Phi$), which leads to the following expression for ${}_3\Phi$:

$$
\begin{aligned}
{}_3\Phi &= \sum_{j=1}^{N} \sum_{k=j}^{N} \sum_{\substack{l=-N\\ l\neq 0}}^{j} \left[{}_3\bar{\Phi}^{(\omega_j+\omega_k+\omega_l)} \, \mathrm{e}^{i(\omega_j+\omega_k+\omega_l)t} \right] \\
&= 6 \sum_{j=1}^{N} \sum_{k=j}^{N} \sum_{\substack{l=-N\\ l\neq 0}}^{j} \left[B_{jkl} \bar{\zeta}_j \bar{\zeta}_k \bar{\zeta}_l \, {}_3\bar{\phi}^{(\omega_j+\omega_k+\omega_l)} \, \mathrm{e}^{i(\omega_j+\omega_k+\omega_l)t} \right]
\end{aligned} \tag{19}
$$

where

$$
{}_3\bar{\Phi}^{(\omega_j+\omega_k+\omega_l)} = 6 B_{jkl} \bar{\zeta}_j \bar{\zeta}_k \bar{\zeta}_l \, {}_3\bar{\phi}^{(\omega_j+\omega_k+\omega_l)}, \tag{20}
$$

$$
\begin{aligned}
B_{jkl} &= \tfrac{1}{6} \quad \text{if} \quad j = k = l, \\
&= \tfrac{1}{2} \quad \text{if} \quad j = k \neq l \quad \text{or} \quad j = l \neq k \quad \text{or} \quad j \neq k = l, \\
&= 1 \quad \text{if} \quad j \neq k \neq l \neq j.
\end{aligned} \tag{21}
$$

Expressions for the boundary conditions on $\bar{S}_b$, $\bar{S}_F$ and S_B, as well as a radiation condition, can now be obtained for each complex potential function ${}_m\bar{\phi}^{(\Sigma\omega)}$, $m = 1, 2, 3$ (see Appendix A).

4. Force calculation

The vertical force on the body can be calculated by integration of the pressure $p - p_0$ over the wetted body surface,

$$
F = \iint_{S_b} (p - p_0) \sin\alpha \, \mathrm{d}S = 2\pi \int_{s_b} (p - p_0) x \, \mathrm{d}x, \tag{22}
$$

s_b being the intersection of S_b with the $0xz$-plane. In order to obtain an expression for the pressure in which values of the potential function on the mean position of the wetted body surface only occur, Bernoulli's equation is expanded in a Taylor series,

$$
\begin{aligned}
p - p_0 = -\rho\big(& g\bar{z} + g\zeta + \Phi_t + \tfrac{1}{2}\Phi_n^2 + \tfrac{1}{2}\Phi_s^2 - \zeta\Phi_{nt} \sin\alpha \\
& + \zeta\Phi_{st}\cos\alpha + \tfrac{1}{2}\zeta^2\Phi_{nnt}\sin^2\alpha - \zeta^2\Phi_{nst}\cos\alpha\sin\alpha + \tfrac{1}{2}\zeta^2\Phi_{sst}\cos^2\alpha \\
& - \zeta\Phi_n\Phi_{nn}\sin\alpha + \zeta\Phi_n\Phi_{ns}\cos\alpha - \zeta\Phi_s\Phi_{ns}\sin\alpha + \zeta\Phi_s\Phi_{ss}\cos\alpha \big).
\end{aligned} \tag{23}
$$

As S_b represents the instantaneous wetted body surface, the pressure has to be integrated over the region on $\bar{S}_b$ between the instantaneous and the mean free-surface positions as well. For the calculation of this part of F, the horizontal coordinate x in (22) is developed into a Taylor series in terms of $\bar{z}$ about the point ($\bar{z}=0$) of $\bar{S}_b$. Several partial integrations lead to an expression in which powers of $(Z-\zeta)$ occur; the latter can be evaluated by calculating the value of $\bar{z}$ for which the series expansion of (23) about ($\bar{z}=0$) equals zero. If (11) is used, the following expression for F is found:

$$F = -2\pi\rho\int_{\bar{S}_b}(g\bar{z}+g\zeta+{}_1\Phi_t)x\,\mathrm{d}x \qquad \text{(1st order)}$$

$$-2\pi\rho\int_{\bar{S}_b}\left[{}_2\Phi_t+\tfrac{1}{2}\left({}_1\Phi_n^2+{}_1\Phi_s^2\right)-\zeta_1\Phi_{nt}\sin\alpha+\zeta_1\Phi_{st}\cos\alpha\right]x\,\mathrm{d}x$$

$$+\left[\pi\rho g\tan\alpha\, r\left(\zeta+\frac{1}{g}{}_1\Phi_t\right)^2\right]_{(\bar{z}=0)} \qquad \text{(2nd order)}$$

$$-2\pi\rho\int_{\bar{S}_b}\left[{}_3\Phi_t+{}_1\Phi_n\,{}_2\Phi_n+{}_1\Phi_s\,{}_2\Phi_s-\zeta_2\Phi_{nt}\sin\alpha+\zeta_2\Phi_{st}\cos\alpha\right.$$

$$+\tfrac{1}{2}\zeta^2{}_1\Phi_{nnt}\sin^2\alpha-\zeta^2{}_1\Phi_{nst}\cos\alpha\sin\alpha$$

$$+\tfrac{1}{2}\zeta^2{}_1\Phi_{sst}\cos^2\alpha-\zeta_1\Phi_n\,{}_1\Phi_{nn}\sin\alpha$$

$$\left.+\zeta_1\Phi_n\,{}_1\Phi_{ns}\cos\alpha-\zeta_1\Phi_s\,{}_1\Phi_{ns}\sin\alpha+\zeta_1\Phi_s\,{}_1\Phi_{ss}\cos\alpha\right]x\,\mathrm{d}x$$

$$+\left[-\tfrac{1}{3}\pi\rho g\left(\zeta+\frac{1}{g}{}_1\Phi_t\right)^3\left(\tan^2\alpha+r\frac{x''-z''\tan\alpha}{\cos^2\alpha}\right)+2\pi\rho\tan\alpha\, r\left(\zeta+\frac{1}{g}{}_1\Phi_t\right)\right.$$

$$\times\left\{{}_2\Phi_t+\tfrac{1}{2}\left({}_1\Phi_n^2+{}_1\Phi_s^2\right)-\zeta_1\Phi_{nt}\sin\alpha+\zeta_1\Phi_{st}\cos\alpha\right.$$

$$\left.\left.-\frac{1}{2\cos\alpha}\left(\zeta+\frac{1}{g}{}_1\Phi_t\right){}_1\Phi_{st}\right\}\right]_{(\bar{z}=0)} \qquad \text{(3rd order)}$$

(24)

where r denotes the radius of the mean waterplane. If use of (1), (12), (14) and (19) is made, it can be shown that the vertical force F can be written as

$$F = {}_1F + {}_2F + {}_3F \qquad (25)$$

with

$${}_1F = \rho g V_0 \sum_{j=1}^{N} \mathrm{Re}\left[{}_1\bar{f}^{(\omega_j)}\frac{\bar{\zeta}_j}{r_0}\,\mathrm{e}^{i\omega_j t}\right], \qquad (26)$$

$$ {}_2F = 2\rho g V_0 \sum_{\substack{k=-N\\k\neq 0}}^{N} \sum_{j=|k|}^{N} A_{jk} \,\mathrm{Re}\left[{}_2\bar{f}^{(\omega_j+\omega_k)} \frac{\bar{\zeta}_j}{r_0} \frac{\bar{\zeta}_k}{r_0} \, \mathrm{e}^{i(\omega_j+\omega_k)t} \right], \tag{27} $$

$$ {}_3F = 6\rho g V_0 \sum_{j=1}^{N} \sum_{k=j}^{N} \sum_{\substack{l=-N\\l\neq 0}}^{j} B_{jkl} \,\mathrm{Re}\left[{}_3\bar{f}^{(\omega_j+\omega_k+\omega_l)} \frac{\bar{\zeta}_j}{r_0} \frac{\bar{\zeta}_k}{r_0} \frac{\bar{\zeta}_l}{r_0} \, \mathrm{e}^{i(\omega_j+\omega_k+\omega_l)t} \right], \tag{28} $$

where V_0 and r_0 are a characteristic volume and length, respectively. General expressions for the nondimensional complex force components ${}_m\bar{f}^{(\Sigma\omega)}$ are given in Appendix B; normal derivatives of potential functions are eliminated making use of Laplace's equation and the boundary conditions on the body.

5. Reduction of order

It seems that the expression for ${}_2\bar{f}^{(\omega_j+\omega_k)}$ contains only one term which requires the knowledge of a second-order potential function; this term can be denoted as

$$ -\frac{2\pi r_0^2}{gV_0} \mathrm{i}\omega_{jk} \int_{\bar{S}_b} {}_2\bar{\phi}^{(\omega_{jk})} x \,\mathrm{d}x \equiv \frac{r_0^2}{gV_0} I_{jk} \tag{29} $$

where

$$ \omega_{jk} = \omega_j + \omega_k. \tag{30} $$

A similar notation can be used for the only term in the expression for ${}_3\bar{f}^{(\omega_j+\omega_k+\omega_l)}$ containing a third-order potential function:

$$ -\frac{2\pi r_0^3}{gV_0} \mathrm{i}\omega_{jkl} \int_{\bar{S}_b} {}_3\bar{\phi}^{(\omega_{jkl})} x \,\mathrm{d}x \equiv \frac{r_0^3}{gV_0} I_{jkl} \tag{31} $$

where

$$ \omega_{jkl} = \omega_j + \omega_k + \omega_l. \tag{32} $$

Expressions for I_{jk} and I_{jkl} can now be found which only contain first-order potentials and first- and second-order potentials, respectively, by means of an order-reduction method, described by Söding [7] for the calculation of second-order forces on a two-dimensional body in a harmonic oscillatory motion. For this purpose, (29) and (31) are written as

$$ I_{jk} = \int\int_{\bar{S}_b} {}_1\bar{\phi}_n^{(\omega_{jk})} {}_2\bar{\phi}^{(\omega_{jk})} \,\mathrm{d}S, \tag{33} $$

$$ I_{jkl} = \int\int_{\bar{S}_b} {}_1\bar{\phi}_n^{(\omega_{jkl})} {}_3\bar{\phi}^{(\omega_{jkl})} \,\mathrm{d}S, \tag{34} $$

where ${}_1\bar{\phi}^{(\omega_{jk})}$ and ${}_1\bar{\phi}^{(\omega_{jkl})}$ denote the linear potential functions for the harmonic motion with frequency $\omega_j + \omega_k$ and $\omega_j + \omega_k + \omega_l$, respectively. Taking into account the boundary conditions on $\bar{S}_F$, S_B and S_R, application of Green's theorem to the functions $({}_1\bar{\phi}^{(\omega_{jk})}, {}_2\bar{\phi}^{(\omega_{jk})})$ and $({}_1\bar{\phi}^{(\omega_{jkl})}, {}_3\bar{\phi}^{(\omega_{jkl})})$, respectively, in a region with boundary $S_T = \bar{S}_F + \bar{S}_b + S_B + S_R$, leads to following expressions for I_{jk} and I_{jkl}:

$$I_{jk} = \int\int_{\bar{S}_b} {}_1\bar{\phi}^{(\omega_{jk})}\, {}_2\bar{\phi}_n^{(\omega_{jk})}\, \mathrm{d}S + \int\int_{\bar{S}_F} {}_1\bar{\phi}^{(\omega_{jk})} \left[\frac{\omega_{jk}^2}{g} {}_2\bar{\phi}^{(\omega_{jk})} - {}_2\bar{\phi}_z^{(\omega_{jk})} \right] \mathrm{d}S, \tag{35}$$

$$I_{jkl} = \int\int_{\bar{S}_b} {}_1\bar{\phi}^{(\omega_{jkl})}\, {}_3\bar{\phi}_n^{(\omega_{jkl})}\, \mathrm{d}S + \int\int_{\bar{S}_F} {}_1\bar{\phi}^{(\omega_{jkl})} \left[\frac{\omega_{jkl}^2}{g} {}_3\bar{\phi}^{(\omega_{jkl})} - {}_3\bar{\phi}_z^{(\omega_{jkl})} \right] \mathrm{d}S. \tag{36}$$

Making use of the boundary conditions for ${}_2\bar{\phi}^{(\omega_{jk})}$ and ${}_3\bar{\phi}^{(\omega_{jkl})}$ on $\bar{S}_b$ and $\bar{S}_F$, these potential functions disappear in expressions (35) and (36), and are replaced by potential functions of lower order. Normal derivatives of potential functions can be eliminated using Laplace's equation and the boundary conditions.

6. Numerical procedure

It appears that calculation of ${}_2\bar{f}^{(\omega_{jk})}$ requires the knowledge of the first-order potential functions ${}_1\bar{\phi}^{(\omega_j)}$, ${}_1\bar{\phi}^{(\omega_k)}$ and ${}_1\bar{\phi}^{(\omega_{jk})}$, while the expression for ${}_3\bar{f}^{(\omega_{jkl})}$ contains the linear potentials ${}_1\bar{\phi}^{(\omega_j)}$, ${}_1\bar{\phi}^{(\omega_k)}$, ${}_1\bar{\phi}^{(\omega_l)}$, ${}_1\bar{\phi}^{(\omega_{jkl})}$ and the second-order potentials ${}_2\bar{\phi}^{(\omega_{jk})}$, ${}_2\bar{\phi}^{(\omega_{jl})}$, ${}_2\bar{\phi}^{(\omega_{kl})}$.

A boundary integration equation method proposed by Yeung [10] (see also Bai and Yeung [1]) and implemented by Kritis [6] (see also Ferdinande and Kritis [4]) for axisymmetric shapes, has been used for the calculation of the linear potential functions, and has been extended in order to obtain the second-order potentials. This method is based on the application of Green's theorem to the functions $1/r$ and $\bar{\phi} \equiv {}_m\bar{\phi}^{(\Sigma\omega)}$ ($m = 1, 2$; $\Sigma\omega = \Sigma_{j=1}^m \omega_j$), r being the distance between a reference point P and a variable point Q, both situated on the boundary S_T:

$$\bar{\phi}(P) = \frac{1}{\psi} \int\int_{S_T} \left[\frac{1}{r} \frac{\partial \bar{\phi}}{\partial n}(Q) - \bar{\phi}(Q) \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right] \mathrm{d}S \tag{37}$$

where ψ is the space angle, equal to 2π if S_T is planar near P, and n denotes the unit normal vector, external with respect to the fluid. S_T is now divided into N_4 discrete axisymmetric elements (see Fig. 2). Boundary integral equation (37) is written N_4 times, considering each discretization point T_i as reference point. If the potential function $\bar{\phi}$ is assumed to be constant for each element, one obtains a linear system of N_4 equations with N_4 variables:

$$2\pi\bar{\phi}_i + \sum_{j=1}^{N_1} \bar{\phi}_j Q_{ij} + \sum_{j=N_1+1}^{N_2} \bar{\phi}_j \left(Q_{ij} - \frac{(\Sigma\omega)^2}{g} P_{ij} \right)$$

$$+ \sum_{j=N_2+1}^{N_3} \bar{\phi}_j \left[Q_{ij} + \left(\frac{1}{2R} + \mathrm{i}k_{(\Sigma\omega)} P_{ij} \right) \right] + \sum_{j=N_3+1}^{N_4} \bar{\phi}_j Q_{ij} = B_i \tag{38}$$

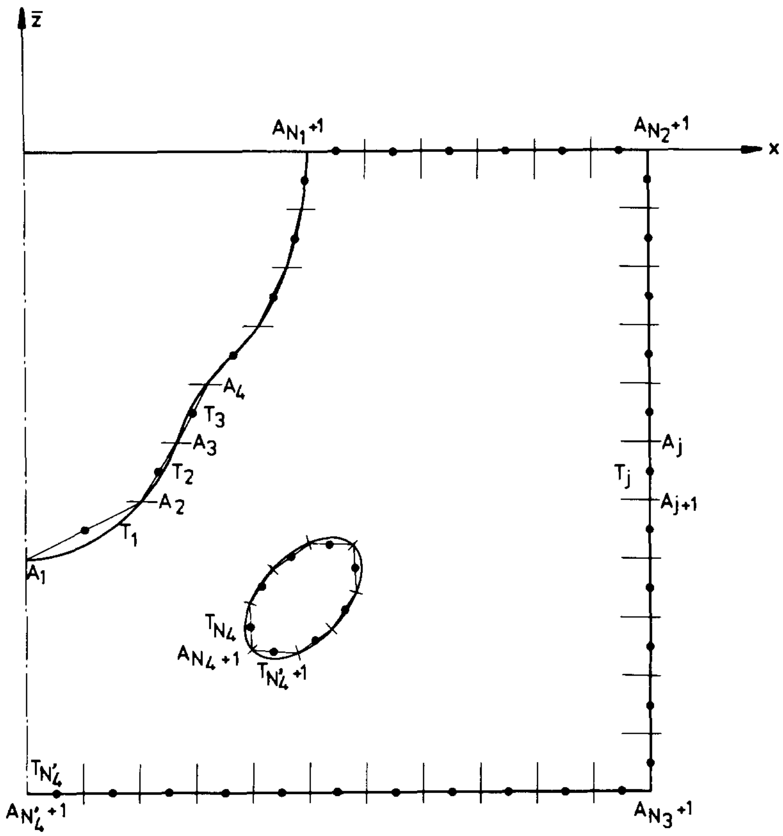

Figure 2.

where

$$B_i = \mathrm{i}\omega \sum_{j=1}^{N_1} P_{ij} \sin \alpha_j, \qquad \text{if} \quad \bar{\phi} = {}_1\bar{\phi}^{(\omega)}, \tag{39a}$$

$$= \tfrac{1}{4} \sum_{j=1}^{N_1} P_{ij} \sin \alpha_j \Big[{}_1\bar{\phi}^{(\omega_k)''} + {}_1\bar{\phi}^{(\omega_l)''} + \frac{1}{x} \sin \alpha \big({}_1\bar{\phi}^{(\omega_k)'} + {}_1\bar{\phi}^{(\omega_l)'} \big)$$

$$- \mathrm{i}\omega_{kl} \Big(\frac{1}{x} \sin \alpha \cos \alpha + z'' \Big) \Big] (T_j)$$

$$- \frac{i}{2g} \sum_{j=N_1+1}^{N_2} P_{ij} \Big[\omega_{kl}\, {}_1\bar{\phi}_x^{(\omega_k)}\, {}_1\bar{\phi}_x^{(\omega_l)} + \frac{1}{2} \frac{\omega_k \omega_l \omega_{kl}}{g} \big(\omega_k^2 + \omega_k \omega_l + \omega_l^2 \big)\, {}_1\bar{\phi}^{(\omega_k)}\, {}_1\bar{\phi}^{(\omega_l)}$$

$$+ \tfrac{1}{2} \omega_k\, {}_1\bar{\phi}^{(\omega_k)} \Big({}_1\bar{\phi}_{xx}^{(\omega_l)} + \frac{1}{x}\, {}_1\bar{\phi}_x^{(\omega_l)} \Big)$$

$$+ \tfrac{1}{2} \omega_l\, {}_1\bar{\phi}^{(\omega_l)} \Big({}_1\bar{\phi}_{xx}^{(\omega_k)} + \frac{1}{x}\, {}_1\bar{\phi}_x^{(\omega_k)} \Big) \Big] (T_j),$$

$$\text{if} \quad \bar{\phi} = {}_2\bar{\phi}^{(\omega_{kl})}, \tag{39b}$$

$$P_{ij} = \iint_{S_j} \frac{1}{r} \, dS, \tag{40}$$

$$Q_{ij} = \iint_{S_j} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS. \tag{41}$$

Expressions for P_{ij} and Q_{ij} are given by Kritis [6], and Ferdinande and Kritis [4]. It is obvious that the systems for calculating ${}_1\bar{\phi}^{(\omega_{kl})}$ and ${}_2\bar{\phi}^{(\omega_{kl})}$ have the same matrix; computing time can be saved if this equality is taken into account.

Numerical calculation of complex force components ${}_m\bar{f}^{(\Sigma\omega)}$ for all combinations of frequencies within the interesting range requires a large amount of computer time. In a more realistic approach, values of ${}_2\bar{f}^{(\omega_j+\omega_k)}$ and ${}_3\bar{f}^{(\omega_j+\omega_k+\omega_l)}$ are calculated for a limited number of particular combinations of frequencies. ${}_2\bar{f}$ and ${}_3\bar{f}$ can be considered as functions of two, respectively three, frequency variables; continuity and symmetry properties make it possible to obtain values for other combinations by interpolation.

The following particular combinations are considered:

for ${}_2\bar{f}^{(\omega_j+\omega_k)}$

$\omega_j =$	ω	ω	ω
$\omega_k =$	ω	$-\omega$	0

for ${}_3\bar{f}^{(\omega_j+\omega_k+\omega_l)}$

$\omega_j =$	ω	ω	ω	ω	ω
$\omega_k =$	ω	ω	ω	$-\omega$	0
$\omega_l =$	ω	$-\omega$	0	0	0

Numerical calculation of these complex force components requires the knowledge of the following potential functions:

$${}_1\bar{\phi}^{(\omega)};\quad {}_1\bar{\phi}^{(2\omega)};\quad {}_1\bar{\phi}^{(3\omega)};$$

$${}_2\bar{\phi}^{(\omega-\omega)};\quad {}_2\bar{\phi}^{(\omega+0)};\quad {}_2\bar{\phi}^{(\omega+\omega)}.$$

Only four different system matrices are needed for the calculation of these potential functions; moreover, the system matrix for ${}_2\bar{\phi}^{(\omega-\omega)}$ is independent of frequency.

7. Conclusion

It has already been suggested (Ferdinande and Vantorre [5]) that the presented radiation theory can be extended to a method leading to the determination of the nonlinear heave response of an axisymmetric system to an irregular sea making use of the "relative motion hypothesis", if the horizontal dimensions of the body are small compared with wave length (long-wave approximation). As this theory concerns a frequency domain approach, transient problems cannot be dealt with, but if only the steady-state response is important, for example when energy absorption from a seaway with a given wave spectrum is considered, application of time-domain methods, which might be more powerful but require large amounts of CPU-time, is avoided. It appears that the presented method does not consume more computer time than required for the solution of the third-order radiation problem for an axisymmetric body in a forced harmonic motion (Vantorre [9]), thanks to the interpolation procedure for determining second- and third-order force components for all frequency combinations starting from the values for a limited number of particular cases.

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Appendix A. Boundary conditions

(i) Laplace's equation

$$
{}_{m}\bar{\phi}_{xx}^{(\Sigma\omega)} + \frac{1}{x}\,{}_{m}\bar{\phi}_{x}^{(\Sigma\omega)} + {}_{m}\bar{\phi}_{zz}^{(\Sigma\omega)} = 0,
$$
$$
\left(m = 1, 2, 3;\ \Sigma\omega = \sum_{j=1}^{m} \omega_j \right). \tag{A.1}
$$

(ii) Free surface conditions ($\bar{S}_F$)

$$
-\omega_j^2\,{}_{1}\bar{\phi}^{(\omega_j)} + g\,{}_{1}\bar{\phi}_z^{(\omega_j)} = 0, \tag{A.2}
$$

$$
-\omega_{jk}^2\,{}_{2}\bar{\phi}^{(\omega_{jk})} + g\,{}_{2}\bar{\phi}_z^{(\omega_{jk})} = \tfrac{1}{2}\mathrm{i} f_{jk}^{(1)}, \tag{A.3}
$$

$$
-\omega_{jkl}^2\,{}_{3}\bar{\phi}^{(\omega_{jkl})} + g\,{}_{3}\bar{\phi}_z^{(\omega_{jkl})} = \tfrac{1}{3}\mathrm{i}\left(\bar{f}_{j,kl}^{(1)} + \bar{f}_{k,jl}^{(1)} + \bar{f}_{l,jk}^{(1)} \right) + \tfrac{1}{24}\left(\bar{f}_{j,kl}^{(2)} + \bar{f}_{k,jl}^{(2)} + \bar{f}_{l,jk}^{(2)} \right) \tag{A.4}
$$

with

$$
\omega_{jk} = \omega_j + \omega_k, \tag{A.5}
$$

$$
\omega_{jkl} = \omega_j + \omega_k + \omega_l, \tag{A.6}
$$

$$
\begin{aligned}
\bar{f}_{jk}^{(1)} = {} & -\omega_{jk}\left({}_{1}\bar{\phi}_x^{(\omega_j)}\,{}_{1}\bar{\phi}_x^{(\omega_k)} + {}_{1}\bar{\phi}_z^{(\omega_j)}\,{}_{1}\bar{\phi}_z^{(\omega_k)} \right) + \tfrac{1}{2}\left(\omega_j\,{}_{1}\bar{\phi}^{(\omega_j)}\,{}_{1}\bar{\phi}_{zz}^{(\omega_k)} + \omega_k\,{}_{1}\bar{\phi}^{(\omega_k)}\,{}_{1}\bar{\phi}_{zz}^{(\omega_j)} \right) \\
& - \frac{1}{2}\,\frac{\omega_j\omega_k}{g}\left(\omega_j\,{}_{1}\bar{\phi}_z^{(\omega_j)}\,{}_{1}\bar{\phi}^{(\omega_k)} + \omega_k\,{}_{1}\bar{\phi}_z^{(\omega_k)}\,{}_{1}\bar{\phi}^{(\omega_j)} \right),
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
\bar{f}_{j,kl}^{(1)} = {} & -\omega_{jkl}\left({}_{1}\bar{\phi}_x^{(\omega_j)}\,{}_{2}\bar{\phi}_x^{(\omega_{kl})} + {}_{1}\bar{\phi}_z^{(\omega_j)}\,{}_{2}\bar{\phi}_z^{(\omega_{kl})} \right) + \tfrac{1}{2}\left(\omega_j\,{}_{1}\bar{\phi}^{(\omega_j)}\,{}_{2}\bar{\phi}_{zz}^{(\omega_{kl})} + \omega_{kl}\,{}_{1}\bar{\phi}_{zz}^{(\omega_j)}\,{}_{2}\bar{\phi}^{(\omega_{kl})} \right) \\
& - \frac{1}{2}\,\frac{\omega_j\omega_{kl}}{g}\left(\omega_j\,{}_{1}\bar{\phi}_z^{(\omega_j)}\,{}_{2}\bar{\phi}^{(\omega_{kl})} + \omega_{kl}\,{}_{1}\bar{\phi}^{(\omega_j)}\,{}_{2}\bar{\phi}_z^{(\omega_{kl})} \right),
\end{aligned} \tag{A.8}
$$

$$\begin{aligned}
\bar{f}^{(2)}_{j,kl} = &-2\,{}_1\bar{\phi}^{(\omega_j)}_{xx}\,{}_1\bar{\phi}^{(\omega_k)}_{x}\,{}_1\bar{\phi}^{(\omega_l)}_{x} - 2\,{}_1\bar{\phi}^{(\omega_j)}_{x}\left({}_1\bar{\phi}^{(\omega_k)}_{z}\,{}_1\bar{\phi}^{(\omega_l)}_{xz} + {}_1\bar{\phi}^{(\omega_l)}_{z}\,{}_1\bar{\phi}^{(\omega_k)}_{xz}\right) \\
&+ {}_1\bar{\phi}^{(\omega_j)}_{zz}\left({}_1\bar{\phi}^{(\omega_k)}_{x}\,{}_1\bar{\phi}^{(\omega_l)}_{x} - {}_1\bar{\phi}^{(\omega_k)}_{z}\,{}_1\bar{\phi}^{(\omega_l)}_{z}\right) \\
&- \frac{\omega_j^2}{g}\,{}_1\bar{\phi}^{(\omega_j)}_{z}\left({}_1\bar{\phi}^{(\omega_k)}_{x}\,{}_1\bar{\phi}^{(\omega_l)}_{x} + {}_1\bar{\phi}^{(\omega_k)}_{z}\,{}_1\bar{\phi}^{(\omega_l)}_{z}\right) + \frac{\omega_k\omega_l}{g}\,{}_1\bar{\phi}^{(\omega_j)}_{zzz}\,{}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)} \\
&- 2\frac{\omega_k\omega_l}{g}\Big[{}_1\bar{\phi}^{(\omega_j)}_{x}\left({}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)}_{xz} + {}_1\bar{\phi}^{(\omega_k)}_{xz}\,{}_1\bar{\phi}^{(\omega_l)}\right) + {}_1\bar{\phi}^{(\omega_j)}_{xz}\left({}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)}_{x} + {}_1\bar{\phi}^{(\omega_k)}_{x}\,{}_1\bar{\phi}^{(\omega_l)}\right) \\
&+ {}_1\bar{\phi}^{(\omega_j)}_{z}\left({}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)}_{zz} + {}_1\bar{\phi}^{(\omega_k)}_{zz}\,{}_1\bar{\phi}^{(\omega_l)}\right) + \tfrac{1}{2}\,{}_1\bar{\phi}^{(\omega_j)}_{zz}\left({}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)}_{z} + {}_1\bar{\phi}^{(\omega_k)}_{z}\,{}_1\bar{\phi}^{(\omega_l)}\right)\Big] \\
&- \frac{\omega_j^2\omega_k\omega_l}{g^2}\Big[{}_1\bar{\phi}^{(\omega_j)}_{zz}\,{}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)} + {}_1\bar{\phi}^{(\omega_j)}_{z}\left({}_1\bar{\phi}^{(\omega_k)}\,{}_1\bar{\phi}^{(\omega_l)}_{z} + {}_1\bar{\phi}^{(\omega_k)}_{z}\,{}_1\bar{\phi}^{(\omega_l)}\right)\Big]. \qquad \text{(A.9)}
\end{aligned}$$

(iii) Body surface conditions ($\bar{S}_b$)

$${}_1\bar{\phi}^{(\omega_j)}_{n} = -\mathrm{i}\omega_j \sin\alpha, \qquad \text{(A.10)}$$

$${}_2\bar{\phi}^{(\omega_{jk})}_{n} = \tfrac{1}{4}\left(\bar{\ell}^{(1)}_j + \bar{\ell}^{(1)}_k\right), \qquad \text{(A.11)}$$

$${}_3\bar{\phi}^{(\omega_{jkl})}_{n} = \tfrac{1}{6}\left(\bar{\ell}^{(1)}_{jk} + \bar{\ell}^{(1)}_{kl} + \bar{\ell}^{(1)}_{jl}\right) - \tfrac{1}{24}\left(\bar{\ell}^{(2)}_j + \bar{\ell}^{(2)}_k + \bar{\ell}^{(2)}_l\right) \qquad \text{(A.12)}$$

with

$$\bar{\ell}^{(1)}_j = {}_1\bar{\phi}^{(\omega_j)}_{nn}\sin\alpha - {}_1\bar{\phi}^{(\omega_j)}_{ns}\cos\alpha, \qquad \text{(A.13)}$$

$$\bar{\ell}^{(1)}_{kl} = {}_2\bar{\phi}^{(\omega_{kl})}_{nn}\sin\alpha - {}_2\bar{\phi}^{(\omega_{kl})}_{ns}\cos\alpha, \qquad \text{(A.14)}$$

$$\bar{\ell}^{(2)}_j = {}_1\bar{\phi}^{(\omega_j)}_{nnn}\sin^2\alpha - 2\,{}_1\bar{\phi}^{(\omega_j)}_{nns}\sin\alpha\cos\alpha + {}_1\bar{\phi}^{(\omega_j)}_{nss}\cos^2\alpha. \qquad \text{(A.15)}$$

(iv) Conditions on bottom and other fixed surfaces (S_B)

$${}_m\bar{\phi}^{(\Sigma\omega)}_{n} = 0 \qquad \left(m = 1, 2, 3;\ \Sigma\omega = \sum_{j=1}^{m}\omega_j\right). \qquad \text{(A.16)}$$

(v) Radiation conditions (S_R)

$${}_m\bar{\phi}^{(\Sigma\omega)}_{x} + \left(\frac{1}{2R} + \mathrm{i}k_{(\Sigma\omega)}\right){}_m\bar{\phi}^{(\Sigma\omega)} = 0 \qquad \text{(A.17)}$$

$$\left(m = 1, 2, 3;\ \Sigma\omega = \sum_{j=1}^{m}\omega_j\right)$$

with

$$k_{(\Sigma\omega)} \tanh k_{(\Sigma\omega)} h = \frac{(\Sigma\omega)^2}{g}. \tag{A.18}$$

Appendix B. Nondimensional complex force components

(i) First-order forces

$$_1\bar{f}^{(\omega_j)} = -\frac{Sr_0}{V_0} - \frac{2\pi r_0}{V_0}\frac{\mathrm{i}\omega_j}{g}\int_{\bar{s}_b} {}_1\bar{\phi}^{(\omega_j)} x \,\mathrm{d}x, \tag{B.1}$$

where S denotes the waterplane area at rest.

(ii) Second-order forces

$$_2\bar{f}^{(\omega_j+\omega_k)} = \frac{-\pi r_0^2}{gV_0}\left\{\int_{\bar{s}_b}\left[2\mathrm{i}\omega_{jk}\,{}_2\bar{\phi}^{(\omega_{jk})} + \tfrac{1}{2}\,{}_1\bar{\phi}^{(\omega_j)\prime}\,{}_1\bar{\phi}^{(\omega_k)\prime} + \tfrac{1}{2}\mathrm{i}\cos\alpha\left(\omega_j\,{}_1\bar{\phi}^{(\omega_j)\prime} + \omega_k\,{}_1\bar{\phi}^{(\omega_k)\prime}\right)\right.\right.$$
$$\left.\left. - \tfrac{1}{2}\sin^2\alpha\left(\omega_j^2 + \omega_j\omega_k + \omega_k^2\right)\right] x \,\mathrm{d}x - \tfrac{1}{2}g\left[r\tan\alpha\; \bar{z}_j\bar{z}_k\right]_{(\bar{z}=0)}\right\} \tag{B.2}$$

where

$$\bar{z}_j = 1 + \frac{\mathrm{i}\omega_j}{g}\,{}_1\bar{\phi}^{(\omega_j)}, \tag{B.3}$$

$$\omega_{jk} = \omega_j + \omega_k. \tag{B.4}$$

(iii) Third-order forces

$$_3\bar{f}^{(\omega_j+\omega_k+\omega_l)} = -\frac{\pi r_0^3}{gV_0}$$
$$\times\left\{2\mathrm{i}\omega_{jkl}\int_{\bar{s}_b}{}_3\bar{\phi}^{(\omega_{jkl})} x\,\mathrm{d}x + \tfrac{1}{3}\int_{\bar{s}_b}\left[{}_1\bar{\phi}^{(\omega_j)\prime}\,{}_2\bar{\phi}^{(\omega_{kl})\prime} + {}_1\bar{\phi}^{(\omega_k)\prime}\,{}_2\bar{\phi}^{(\omega_{jl})\prime} + {}_1\bar{\phi}^{(\omega_l)\prime}\,{}_2\bar{\phi}^{(\omega_{jk})\prime}\right.\right.$$
$$\left. + \mathrm{i}\cos\alpha\left(\omega_{kl}\,{}_2\bar{\phi}^{(\omega_{kl})\prime} + \omega_{jl}\,{}_2\bar{\phi}^{(\omega_{jl})\prime} + \omega_{jk}\,{}_2\bar{\phi}^{(\omega_{jk})\prime}\right)\right] x\,\mathrm{d}x$$

$$+\frac{i}{12}\int_{\bar{s}_b}\Big(\big(\omega_j+\omega_{kl}\sin^2\alpha\big)_1\bar{\phi}^{(\omega_j)''}+\big(\omega_k+\omega_{jl}\sin^2\alpha\big)_1\bar{\phi}^{(\omega_k)''}$$

$$+\big(\omega_l+\omega_{jk}\sin^2\alpha\big)_1\bar{\phi}^{(\omega_l)''}\Big)x\,\mathrm{d}x$$

$$+\tfrac{1}{12}\mathrm{i}\omega_{jkl}\int_{\bar{s}_b}\big({}_1\bar{\phi}^{(\omega_j)'}+{}_1\bar{\phi}^{(\omega_k)'}+{}_1\bar{\phi}^{(\omega_l)'}\big)\sin^3\alpha\,\mathrm{d}x$$

$$+\frac{i}{12}\int_{\bar{s}_b}\big(\omega_{kl\,1}\bar{\phi}^{(\omega_j)'}+\omega_{jl\,1}\bar{\phi}^{(\omega_k)'}+\omega_{jk\,1}\bar{\phi}^{(\omega_l)'}\big)xx''\sin\alpha\,\mathrm{d}x$$

$$+\tfrac{1}{12}\int_{\bar{s}_b}\Big[{}_1\bar{\phi}^{(\omega_j)'}\big({}_1\bar{\phi}^{(\omega_k)''}+{}_1\bar{\phi}^{(\omega_l)''}\big)+{}_1\bar{\phi}^{(\omega_k)'}\big({}_1\bar{\phi}^{(\omega_j)''}+{}_1\bar{\phi}^{(\omega_l)''}\big)$$

$$+{}_1\bar{\phi}^{(\omega_l)'}\big({}_1\bar{\phi}^{(\omega_j)''}+{}_1\bar{\phi}^{(\omega_k)''}\big)\Big]\cos x\,\mathrm{d}x$$

$$+\tfrac{1}{12}\omega_{jkl}^2\int_{\bar{s}_b}(\cos\alpha\sin\alpha+xz'')\sin^2\alpha\,\mathrm{d}x$$

$$-\tfrac{1}{12}\big(\omega_j^2+\omega_k^2+\omega_l^2\big)\int_{\bar{s}_b}\cos\alpha\sin\alpha\,xx''\,\mathrm{d}x$$

$$+\Big[\tfrac{1}{12}g\Big(\tan^2\alpha+r\frac{x''-z''\tan\alpha}{\cos^2\alpha}\Big)\bar{z}_j\bar{z}_k\bar{z}_l$$

$$-\tfrac{1}{3}r\tan\alpha\,\mathrm{i}\big(\omega_{kl}\bar{z}_{j\,2}\bar{\phi}^{(\omega_{kl})}+\omega_{jl}\bar{z}_{k\,2}\bar{\phi}^{(\omega_{jl})}+\omega_{jk}\bar{z}_{l\,2}\bar{\phi}^{(\omega_{jk})}\big)$$

$$-\tfrac{1}{12}r\tan\alpha\Big\{-\big(\omega_j^2+\omega_j\omega_k+\omega_k^2\big)\bar{z}_j-\big(\omega_j^2+\omega_j\omega_l+\omega_l^2\big)\bar{z}_k$$

$$-\big(\omega_k^2+\omega_k\omega_l+\omega_l^2\big)\bar{z}_l+\bar{z}_{j\,1}\bar{\phi}^{(\omega_k)'}{}_1\bar{\phi}^{(\omega_l)'}$$

$$+\bar{z}_{k\,1}\bar{\phi}^{(\omega_j)'}{}_1\bar{\phi}^{(\omega_l)'}+\bar{z}_{l\,1}\bar{\phi}^{(\omega_j)'}{}_1\bar{\phi}^{(\omega_k)'}$$

$$+\mathrm{i}\cos\alpha\Big[\omega_{j\,1}\bar{\phi}^{(\omega_j)'}(\bar{z}_k+\bar{z}_l)+\omega_{k\,1}\bar{\phi}^{(\omega_k)'}(\bar{z}_j+\bar{z}_l)$$

$$+\omega_{l\,1}\bar{\phi}^{(\omega_l)'}(\bar{z}_j+\bar{z}_k)\Big]\Big\}\Big]_{(\bar{z}=0)}\Bigg\}\qquad\text{(B.5)}$$

where

$$\omega_{jkl}=\omega_j+\omega_k+\omega_l.\qquad\text{(B.6)}$$

References

[1] K.J. Bai and R.W. Yeung, Numerical solution to free-surface flow problems, *Proceedings 10th Symposium on Naval Hydrodynamics*, Cambridge, Mass. (1974) 609–33.

[2] V. Ferdinande, A study of wave energy absorption by floating bodies (in Dutch), Dienst voor Scheepsbouwkunde R.U.G. Rep. 4–82 & 11–82, Gent (1982).

[3] V. Ferdinande, On power absorption by floating devices in waves, *International Shipbuilding Progress* 32, 373 (1985) 204–209.

[4] V. Ferdinande and B.G. Kritis, An economical method of determining added mass and damping coefficients of axisymmetric floating bodies in oscillatory heaving motion, *International Shipbuilding Progress* 27, 313 (1980) 231–240.

[5] V. Ferdinande and M. Vantorre, The concept of a bipartite point absorber, in: IUTAM Symposium (1985: Lisbon, Portugal) – *Hydrodynamics of ocean wave-energy utilization*, Editors: D.V. Evans and A.F. de O. Falcão, Springer-Verlag, Berlin, Heidelberg (1986) 217–226.

[6] B.G. Kritis, A fast theoretical and an experimental method of determining the hydrodynamic forces on axisymmetric floating bodies in oscillatory heaving motion, Doctor's Thesis R.U.G., Gent (1979).

[7] H. Söding, Second-order forces on oscillating cylinders in waves, *Schiffstechnik* 23, 114 (1976) 205–209.

[8] M. Vantorre, Third-order potential theory for the determination of hydrodynamic forces on axisymmetrical floating and immersed bodies in forced periodic heaving (in Dutch), Doctor's Thesis R.U.G., Gent (1985).

[9] M. Vantorre, Third-order theory for determining the hydrodynamic forces on axisymmetric floating or submerged bodies in oscillatory heaving motion, to be published in *Ocean Engineering* (1986).

[10] R.W. Yeung, A singularity-distribution method for free-surface flow problems with an oscillating body, College of Engineering Rep. NA 73-6, University of California, Berkeley (1973).